A SOUSLIN OPERATION FOR Π^1_2

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ABSTRACT

Throughout this paper we assume the existence of a measureable cardinal. Membership in a Π^1_2 set of reals is shown to be equivalent to the existence of an infinite path in a tree $\mathscr{T}^*$ of pairs of finite sequences of natural numbers and ordinals. This is used to prove that every Π^1_2 relation can be uniformized by a Π^1_2 function. One gets that under certain assumptions the Shoenfield Absoluteness Theorem holds for Σ^1_3 statements, that Π^1_3 sets include perfect subsets and that Σ^1_3 sets are Lebesgue measureable and have the Baire Property.

Many theorems of descriptive set theory arise from the fact that various classes of sets can be represented as the result of applying the Souslin operation to systems of clopen sets in various topological spaces. The situation is quite simply that where the Souslin operation can be applied there is a wealth of known results, and where it cannot there is very little known. The classical example is the analytic sets. More recently it was discovered that the Souslin operation could also be applied to the Π^1_1 sets with many fundamental results, e.g. the Kondo-Addison uniformization theorem or the Shoenfield absoluteness lemma for Σ^1_2 [12, p. 185] [13]. In this case the innovation is to use for the basic topological space not the Baire space but rather the product space $\aleph_1^N$. (Where $\aleph_1$ is the first uncountable cardinal and N is the set of non-negative integers.) In this paper we shall show this same operation can be applied to the Π^1_2 sets by using the first measureable cardinal instead of $\aleph_1$.

The fundamental idea is that functions from the measurable cardinal κ into the ordinals represent ordinals in the ultraproduct of the universe of set theory. Since the ultraproduct is well-founded, this enables us to Gödel number long

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sequences of ordinals with single ordinals. This Gödel numbering possibility was first noticed by Solovay and Martin [8]. What we do here grows out of that paper and started as an attempt to clear up a technical point which they left open. In doing so we are able, among other things, to improve their result. They prove that every nonempty Π^1_2 set of reals contains a Δ^1_4 real. We shall show that every non-empty Π^1_2 set includes a Π^1_3 singleton.

I suspect that sometime in the near future someone will be able to push the construction which is begun here through the entire 2nd order hierarchy and show that every non-empty Π^1_{2n+1} set contains a Π^1_{2n+1} singleton. All I can say for certain however is that I have worked very hard on this point, and the whole process always hangs up on the same seemingly fragile but so far immovable obstacle.

Throughout the paper we shall try to use standard notation wherever possible. Small Greek letters shall be used to represent functions from N into N and shall be loosely referred to as real numbers. We assume that the reader has adopted some standard recursive coding of finite sequences of integers by single integers, so that when we refer to the sequence number s we shall always mean the number which codes the finite sequence s. The length of the sequence coded by the number s will be given by the function $1h(s)$. We assume that the reader's coding system satisfies the condition that $1h(s) \leqq s$ for every number s. Some of the constructions depend very heavily on the fact that sequence numbers can be regarded simultaneously as both sequences and as numbers. If α is a number theoretic function, $\bar{\alpha}(n)$ denotes the number of the sequence $\langle \alpha(0), \cdots, \alpha(n) \rangle$. Sometimes we will be dealing with a function f from N into some other set, then $\bar{f}(n)$ will just be the sequence $\langle f(0), \cdots, f(n) \rangle$.

A tree is a set of sequence numbers such that whenever a given sequence number is in the set all of its initial subsequence numbers are also there. We shall express the relation s' is an initial subsequence of s by $s' \subset s$. A path through the tree S will be a function α such that for every n, $\bar{\alpha}(n)$ is in S. Sometimes the tree will consist of pairs or triples of sequence numbers of the same length.

Trees are given a partial ordering relation; we say that a sequence is *smaller* than any of its initial subsequences. This partial ordering extends to the familiar Kleene-Brouwer lexicographical linear ordering. In this ordering any two sequences which are incomparable in the above partial ordering are ordered by first difference as in a dictionary. That sequence which is smaller at the first place at which the two sequences differ is the smaller sequence.

1. Π_2^1 sets as the result of the Souslin operation

We start with a Π_2^1 set A. That is to say we start with a set of reals A and a tree T of triples of sequences of integers satisfying,

$$\alpha \in A \equiv \forall \beta \exists \gamma \, \forall n[\langle \bar{\alpha}(n), \bar{\beta}(n), \gamma(n) \rangle \in T].$$

Then, proceeding along well tried lines, define

$$T^{\alpha,\beta} =_{def} \{u \colon \exists n[\langle \bar{\alpha}(n), \bar{\beta}(n), u \rangle \in T]\}$$

and rewrite the definition of A as,

$$\alpha \in A \equiv \forall \beta [T^{\alpha,\beta} \text{is not well-ordered}]$$

where the order relation is the Kleene-Brouwer lexicographical ordering. Let us make the further definition that for sequence numbers s, t of the same length,

$$T^{s,t} = \{u \colon u < 1h(s) \wedge \exists s', t'[s' \subset s \wedge t' \subset t \wedge \langle s', t', u \rangle \in T]\}$$

remembering that we are using sequence numbers so that $T^{s,t}$ is just a finite set of numbers bounded in the natural ordering by the number $1h(s)$. Clearly,

$$T^{\alpha,\beta} = \bigcup_{n<\omega} T^{\bar{\alpha}(n),\bar{\beta}(n)}.$$

Also, as a matter of convenience let $f \colon G \to H$ mean that f is a morphism from G to H in the category of ordered sets. In the special case where G is a tree let us stipulate that f must return a zero for all those sequences not in G. (Zero will always be a member of H). Conversely, $f \colon G \nrightarrow H$ means that the function f is *not* such a morphism. Then letting κ be the first measurable cardinal, the definition of A can be altered still further:

$$\begin{aligned} \alpha \in A &\equiv \forall \beta \forall f[f \colon T^{\alpha,\beta} \nrightarrow \kappa] \\ &\equiv \forall \beta \forall f \, \exists n[\bar{f}(n) \colon T^{\bar{\alpha}(n),\bar{\beta}(n)} \nrightarrow \kappa]. \end{aligned}$$

It may seem that the use of κ here is extravagantly generous since any uncountable ordinal will do, but it will be seen that this choice is well justified.

Since the above predicate has something of the form of a Π_1^1 predicate, it seems reasonable to consider its unsecured sequences. Accordingly,

$$\mathfrak{J} = \{\langle s, t, u \rangle \colon 1h(u) = 1h(s) = 1h(t) \wedge u \colon T^{s,t} \to \kappa\}.$$

$$\mathfrak{J}^{\alpha} = \{\langle t, u \rangle \colon \exists n[\langle \bar{\alpha}(n), t, u \rangle \in \mathfrak{J}]\}.$$

One remark that we shall need later is the following: If $\langle s, t, u \rangle \in \mathfrak{J}$, then the ordinal sequence u is zero at those sequence numbers $< 1h(s)$ which are not in $T^{s,t}$ but apart from this proviso, the proposition $\langle s, t, u \rangle \in \mathfrak{J}$ depends only on the order relationships holding between the various entries of u and has no dependence on the actual ordinals used. This given s, t and the range of the sequence u we can uniquely recapture u as the unique sequence of the appropriate length with that particular range which gives an element of $\mathfrak{J}$ (Uniqueness is given by the linearity of the ordering). This remark will be used when we come to putting a measure on $\{u: \langle s,t,u\rangle \in \mathfrak{J}\}$. In $\mathfrak{J}$ and hereafter we shall use only the tree partial ordering and not the Kleene-Brouwer lexicographical ordering because this uniqueness will not be required.

Again we can seemingly eliminate quantifiers from the definition of A.

$$\alpha \in A \equiv \forall \beta \forall f \exists n \langle \bar{\beta}(n), \bar{f}(n)\rangle \notin \mathfrak{J}^\alpha$$
$$\equiv \mathfrak{J}^\alpha \text{ is well-founded}$$

Just as before let us convert the universal statement $\mathfrak{J}^\alpha$ is well-founded into an existential statement by means of ordinal valued functions. That is, letting κ^+ be the first cardinal larger that κ,

$$\alpha \in A \equiv \exists f [f : \mathfrak{J}^\alpha \to \kappa^+].$$

Simplifying this last formula into something involving finite sequences is where we begin to encounter some difficulties. The problem is that the obvious space from which the finite sequences should be taken is not an ordinal but rather a set of ordinal valued functions. However, as a first baby step let us use these.

Proceeding just as before, let

$$\mathfrak{J}^s =_{def} \{\langle t, u\rangle : t < lh(s) \subset \exists s' \subset s[\langle s', t, u\rangle \in \mathfrak{J}]\}$$

Given a morphism f from $\mathfrak{J}^\alpha$ into κ^+ we note that it is a function of two variables so that we may agree to write the integer variable t as a subscript and write f as an infinite sequence, $\langle f_t\rangle$. Let $\bar{f}(n)$ denote the sequence $\langle f_0, \cdots, f_n\rangle$. Then,

$$\alpha \in A \equiv \exists f \forall n [\bar{f}(n) : \mathfrak{J}^{\bar{\alpha}(n)} \to \kappa^+].$$

Again taking the tree of unsecured sequences, we define

$$\mathfrak{J}^+ = \{\langle s, v\rangle : 1h(s) = 1h(v) \wedge v : \mathfrak{J}^s \to \kappa^+\}.$$

Since this definition plays a central role in all further development we pause a moment to discuss the conventions implicit in this use of the arrow notation. If for each k less than $1h(v)$ we let v_k be the kth entry of v, the above notation requires that for each such k v_k is a function from $\{u: \langle k, u\rangle \in \mathfrak{J}^s\}$ into κ^+ and that if $\langle k, u\rangle$ is a initial subsequence of $\langle k', u'\rangle$ and $\langle k', u'\rangle$ is in

$\mathfrak{J}^s$, then $v_{k'}(u') < v_k(u)$. It is quite easily seen that if s is an initial subsequence of s', $\mathfrak{J}^{s'} \cap \{\langle t, u\rangle : t \leqq 1h(s)\} = \mathfrak{J}^s$, and thus if $\langle s', v'\rangle$ is in $\mathfrak{J}^+$ so is $\langle s, v' \restriction 1h(s)-1\rangle$. We can now write,

$$\alpha \in A \equiv \exists f\, \forall n[\langle \bar{\alpha}(n), \bar{f}(n)\rangle \in \mathfrak{J}^+].$$

We have so far done nothing which at this point in time can be considered unusual, however our next task of converting the sequences v into ordinal sequences adds some interest to the construction. In order to do this we must return to the fact that κ was chosen to be a measurable cardinal. The facts which we shall state about the measurable cardinal can all be found in the literature and will not be proven here.

If μ is a *normal* measure on κ and if for any set x we let $x^{[n]}$ denote the collection of n-element subsets of x, a two-valued measure μ_n can be defined on $\kappa^{[n]}$ as follows:

$$\mu_n(y) = 1 \equiv_{def} \exists x[x \subseteq \kappa \wedge \mu(x) = 1 \wedge x^{[n]} \subseteq y].$$

The fact that this definition does indeed give a measure on $\kappa^{[n]}$ is proven by a nontrivial inductive argument [8, p. 142]. These measures μ_n can be transferred to the tree $\mathfrak{J}$. As we noted in our remarks immediately following the definition of $\mathfrak{J}$, the proposition $\langle s, t, u\rangle \in \mathfrak{J}$ depends *only* on the order relationships which hold between the various entries of u. If we let $\#(s, t)$ be the cardinality of $T^{s,t}$, this means that any set x in $\kappa^{[\#(s\,t)]}$ can be arranged in one and only one way such that with addition of the appropriate zeros the resulting sequence u will be such that the triple $\langle s, t, u\rangle$ is in $\mathfrak{J}$. This one-to-one correspondence gives a measure $\mu_{s,t}$ on $\{u : \langle s, t, u\rangle \in \mathfrak{J}\}$. This measure can be defined directly in exactly the same manner as the measure $\mu_{\#(s,t)}$. That is to say, a set y is of $\mu_{s,t}$ measure one iff there is a measure one subset x of κ such that whenever all the non-zero elements of a sequence u are taken from x and $\langle s, t, u\rangle \in \mathfrak{J}$, then u must necessarily be in y.

We now define a partial ordering on the set of all ordinal valued functions on $\{u : \langle s, t, u\rangle \in \mathfrak{J}\}$. Namely,

$$f < g \equiv_{def} \mu_{s\,t}\{u : f(u) < g(u)\} = 1.$$

It is the fundamental fact of the theory of measurable cardinals that this ordering is in fact *well-founded* [8, p. 140]. If in addition we factor our function space by the equivalence relation "equal almost everywhere," it becomes a linear well-ordering [8, p. 140]. Let $\lambda(f)$ be the ordinal of f in the above ordering, then

$$\lambda(f) = \lambda(g) \equiv f = g \text{ almost everywhere.}$$

This brings us to the central definition of this paper. For $v = (v_0, \cdots, v_n)$ a sequence of functions in the domain of λ let us denote the sequence $\langle \lambda(v_0), \cdots, \lambda(v_n) \rangle$ by the symbol $\lambda(v)$.

DEFINITION. $\mathfrak{J}^* = \{\langle s, \lambda(v) \rangle : \langle s, v \rangle \in \mathfrak{J}^+\}$.

More correctly, of course, we should use some such symbol as $\mathfrak{J}^*_A$ because the original Π^1_2 set A has been carried as a parameter throughout the construction. But we will save this exactitude until it is needed.

THEOREM 1. $\alpha \in A \equiv \exists f \, \forall n \langle \bar{\alpha}(n), \bar{f}(n) \rangle \in \mathfrak{J}^*$.

One implication is straightforward for $\alpha \in A$ implies that $(\mathfrak{J}^+)^\alpha$ $(= \{v : \exists n(\langle \bar{\alpha}(n), v \rangle \in \mathfrak{J}^+)\})$ is not well-founded and the function λ converts any path through this tree into a path through $(\mathfrak{J}^*)^\alpha$ $(= \{w : \exists n(\langle \bar{\alpha}(n), w \rangle \in \mathfrak{J}^*)\})$. For the converse implication we shall prove somewhat more than has so far been stated. The reason for this is that it is convenient to prove Theorem 2 simultaneously with Theorem 1.

The set of ordinal valued functions on the integers can be linearly ordered by first difference. Clearly the set of paths through any tree has a minimal element in this ordering.

THEOREM 2. *If there is a function f such that $\langle \alpha, f \rangle$ is a path through $\mathfrak{J}^*$ then $\alpha \in A$. Furthermore, if f is the smallest such path it is the sequence of ordinals which represents the height functions on the well-founded tree $\mathfrak{J}^\alpha$.*

PROOF. Given a path $\langle \alpha, f \rangle$ through $\mathfrak{J}^*$ we know that for each n there is a function sequence v^n of length $n + 1$ such that $\langle \bar{\alpha}(n), v^n \rangle \in \mathfrak{J}^+$ and $\lambda(v^n) = \bar{f}(n)$. It would be tempting to write $\bar{v}(n)$ instead of v^n for this sequence but that would be incorrect since we do not know that for $n \neq m$ v^n and v^m are comparable sequences. We let v^n_k be the kth entry of v^n, then v^n is a function from $\{u : \langle \bar{\alpha}(1h(k) - 1), k, u \rangle \in \mathfrak{J}\}$ into κ^+ and for every n, m and $k \leqq \min\{n, m\}$, $\lambda(v^n_k) = \lambda(v^m_k) = f(k)$. Thus there is a subset $D_{n,m,k}$ of κ with μ-measure one such that on all those ordinal sequences u which use ordinals only from $D_{n,m,k}$, v^n_k is equal to v^m_k. Since all of the countably many $D_{n,m,k}$ are of μ-measure one their intersection D is also of μ-measure one. Let us use the notation x^∞ for the set of all finite sequences from the set x. We define $v'_k = v^{k+1}_k \upharpoonright D^\infty$. Then for every $n > k$ $v^n_k \upharpoonright D^\infty = v'_k$ and we may define a function v by the equation $v(k, u) = v'_k(u)$.

It follows directly that $v:\mathfrak{J}^{\alpha}\cap(N\times D)^{\infty}\to\kappa^{+}$. Now let H be an order isomorphism from κ onto D. H clearly induces a morphism H^* from $\mathfrak{J}^{\alpha}$ into $(N\times D)^{\infty}$. Namely,

$$H^*(\langle t,\langle\tau_1,\cdots,\tau_n\rangle\rangle)=\langle t,\langle H(\tau_1),\cdots,H(\tau_n)\rangle\rangle.$$

But, as we have remarked on two previous occasions, the proposition $\langle t,u\rangle\in\mathfrak{J}^{\alpha}$ depends only on the order relationships which hold between the various entries of u. Thus H^* is in fact a *tree isomorphism from* $\mathfrak{J}^{\alpha}$ *onto* $\mathfrak{J}^{\alpha}\cap(N\times D)^{\infty}$. So the composition of the two functions H^* and v is a morphism from $\mathfrak{J}^{\alpha}$ into the ordinals. Hence $\mathfrak{J}^{\alpha}$ is well-founded; concluding the proof of Theorem 1.

However, let us do a bit more diagram chasing to get the extra information of Theorem 2. H is a strictly increasing function with a measure one range. It follows directly from the definition of a normal measure that any such function has a measure one set of fixed points [8, p. 141]. One more bit of notation: For any well-founded tree S and any node s in S let $\|S\|_s$ be the height function on S evaluated at s. Then, since any morphism from any well-founded tree majorizes the height function [8, p. 185],

$$\|\mathfrak{J}^{\alpha}\|_{\langle t,u\rangle}\leqq v(H^*(t,u)).$$

But, if we let E be the measure one set of fixed points for H, and u any sequence from E (of the appropriate length),

$$H^*(t,u)=\langle t,u\rangle$$

and thus, for u any sequence from E,

$$\|\mathfrak{J}^{\alpha}\|_{\langle t,u\rangle}\leqq v_t(u).$$

In other words, $\|\mathfrak{J}^{\alpha}\|_{[t,\cdot]}$ is less than or equal to v_t almost everywhere, and

$$\lambda(\|\mathfrak{J}^{\alpha}\|_{\langle t,\cdot\rangle})\leqq\lambda(v_t)=f(t)$$

which concludes the proof of Theorem 2.

2. The ordinal definability of $\mathfrak{J}^*$

For some applications we will need to know that $\mathfrak{J}^*$ is an ordinal definable set. As things stand right now that need not be the case. It has only been defined in terms of a normal measure and Kunen and Paris have independently given a model for set theory in which there are no definable measures [10]. The situation is brightened by the fact, also proven by both Kunen and Paris, that in L_{μ}, the smallest transitive model for set theory in which κ is a measurable cardinal, the normal measure is unique and hence definable [10]. (The definition of L_μ which I have just given is not quite the usual one. Normally L_μ is defined as the sets hereditarily constructible from some fixed measure μ [14]. But again Kunin and Paris show that this definition is in fact independent of μ and is equivalent to the definition I gave [10].)

Our problem is slightly complicated by the fact that L_μ does not necessarily contain all the real numbers. However, the Kunin-Paris arguments still work if instead of L_μ we use M_μ, the least transitive model for set theory containing all real numbers and in which κ is still measureable. So in M_μ, $\mathfrak{J}^*$ is ordinal definable, and furthermore we have just defined M_μ. So the object we get by relativizing our definition of $\mathfrak{J}^*$ to M_μ is ordinal definable in the universe. Since M_μ contains all reals we may clearly use this new $\mathfrak{J}^*$ in place of the old.

STIPULATION. $\mathfrak{J}^*$ is ordinal definable.

3. Applications

In this section we shall use Theorems 1 and 2 to prove analogs of the Kondo-Addison uniformization theorem [12, p. 185] and the Shoenfield absoluteness Lemma [13]. We shall then indicate how these can be used to establish results about the perfect subsets of or Lebesgue measurability of Σ^1_3 sets.

The Kondo-Addison uniformization theorem says that any Π^1_1 relation can be uniformized by a Π^1_1 function. That is, if $\phi(\alpha,\beta)$ is a Π^1_1 relation, there is a Π^1_1 partial function which for each α picks out a β satisfying the relation ϕ. We shall show that any Π^1_2 relation can be uniformized by a Π^1_3 function. Most of the work for this theorem has already been done in Theorem 2, but we have broken the proof into two parts in order to emphasize the construction of the tree $\mathfrak{J}^*$ as a separate entity.

Briefly, what must be done is to give a uniform method of selecting a member of a Π^1_2 set and then to prove the definability of this method. By now the selection method is self-evident. Namely, in the lexicographical ordering the set of paths through the tree $\mathfrak{J}^*$ has a minimal element $\langle \alpha_0, f_0 \rangle$; α_0 is the selected real. We now show that $\{\alpha_0\}$ is a Π^1_3 set. First let us define a function λ^* by the equation

$$\lambda^*(\alpha, t) = \lambda(\|\mathfrak{J}^\alpha\|_{\langle t,\cdot\rangle})$$

Theorem 2 says that for every integer t, $\lambda^*(\alpha_0, t) = f_0(t)$.

LEMMA. *The two predicates $\lambda^*(\alpha,t) = \lambda^*(\beta,t)$ and $\lambda^*(\alpha,t) < \lambda^*(\beta,t)$ are both Δ^1_3 in the three variables α, β, t.*

We shall only give a quick sketch of the proof of this lemma. It is done in detail in Martin-Solovay [8, p. 148]. The basic idea is that each is equivalent to saying that something is true in $L(\alpha,\beta)$ and so, by the Silver-Solovay results with undiscernables [14], [16], is equivalent to certain Gödel number being in the Δ^1_3 set $\langle\alpha,\beta\rangle^{\#}$. We give $\lambda^*(\alpha,t)=\lambda^*(\beta,t)$ as an example. This is equivalent to saying that for almost all sequences u,

$$\|\mathfrak{J}^{\alpha}\|_{\langle t,u\rangle}=\|\mathfrak{J}^{\beta}\|_{\langle t,u\rangle}.$$

But both $\mathfrak{J}^{\alpha}$ and $\mathfrak{J}^{\beta}$ are in $L(\alpha,\beta)$, thus this equality can be expressed in the form

$$L(\alpha,\beta)\ \models\phi(\kappa,\alpha,\beta,t,u)$$

where ϕ is some particular formula of set theory. The parameter κ creeps in because of its role in defining the tree $\mathfrak{J}$. We now use the two crucial facts that

1.) The set of cardinals is of measure one.

2.) The set of cardinals is a set of indiscernables for $L(\alpha,\beta)$.

These give that for some integer n, depending on t,

$$\begin{aligned}\lambda^*(\alpha,t) &= \lambda^*(\beta,t)\\ &\equiv L(\alpha,\beta)\ \models\psi(\kappa,\alpha,\beta,t,\aleph_1,\cdots,\aleph_n)\\ &\equiv L(\alpha,\beta)\ \models\psi(\aleph_{n+1},\alpha,\beta,t,\aleph_1,\cdots,\aleph_n)\\ &\equiv \text{"}\psi(c_{n+1},\alpha,\beta,t,c_1,\cdots,c_n)\text{"}\in\langle\alpha,\beta\rangle^{\#}\end{aligned}$$

where again ψ is a formula of set theory which we have chosen not to explicitly write down. "ψ" means the Gödel number of ψ (which for us is ψ itself). The second equivalence follows from the happy circumstance that as far as $L(\alpha,\beta)$ is concerned κ is just as indiscernable as any other cardinal, and so we may erase the κ of line 2 and insert the $\aleph_{n+1}$ of line 3. This completes the proof of the lemma.

We now give the number theoretic definition of $\{\alpha_0\}$. We proceed by a simultaneous inductive definition (recalling that A is our original Π^1_2 set).

$$\begin{aligned}A_0 &= A\\ n_k &= \text{the least member of } \{\alpha(k):\alpha\in A_k\}.\\ B_k &= A_k\cap\{\alpha:\alpha(k)=n_k\}\\ \sigma_k &= \text{the least member of } \{\lambda^*(\alpha,k):\alpha\in B_k\}.\\ A_{k+1} &= B_k\cap\{\alpha:\lambda^*(\alpha,k)=\sigma_k\}\end{aligned}$$

Clearly, $n_k = \alpha_0(k)$ and $\sigma_k = f_0(k)$ and $\{\alpha_0\} = \cap_k A_k$. We need now only count quantifiers. For any α in A_k, α is *not* in A_{k+1} iff

$$\begin{aligned}\exists\beta[\bar\beta(k-1) &= \bar\alpha(k-1) \wedge \bar\lambda^*(\beta, k-1) = \bar\lambda^*(\alpha, k-1)\\ &\wedge [\beta(k) < \alpha(k) \vee [\beta(k) = \alpha(k)\\ &\wedge \lambda^*(\beta, k) < \lambda^*(\alpha, k)]]]\end{aligned}$$

From the lemma, the matrix of this formula is Δ^1_3 and so

$$\alpha \in \bigcap_k A_k \equiv \alpha \in A \wedge \forall k\, \forall \beta \neg R(\alpha, \beta, k)$$

where R is the above Δ^1_3 matrix. We have completed the proof of:

THEOREM 3. *Any Π^1_2 relation can be uniformized by a Π^1_3 function (assuming the existence of a measurable cardinal).*

One way of stating the Shoenfield absoluteness lemma is "If ϕ is any Σ^1_2 Formula and M is any transitive model for set theory containing all ordinals, then ϕ is equivalent to $\phi^{(M)}$, where $\phi^{(M)}$is the relativization of ϕ to the model M." In the following theorems we generalize this statement to Σ^1_3.

THEOREM 4. *If $\phi(\alpha)$ is any Σ^1_3 statement in the parameter α, and M is any transitive model for set theory containing all ordinals, the real number α, and the tree $\mathfrak{J}^*$ associated with ϕ, then ϕ is equivalent to $\phi^{(M)}$.*

PROOF. We write $\phi(\alpha)$ in the form $\exists\beta[\langle\alpha,\beta\rangle \in A]$ where A is a Π^1_2 set. Then in view of the Shoenfield Lemma we need only show that $\phi(\alpha)$ is equivalent to $\exists\beta[\beta \in M \wedge \langle\alpha,\beta\rangle \in A]$. But since both $\mathfrak{J}^*$ and α are in M, $(\mathfrak{J}^*)^\alpha$ is also in M. Now since M contains all ordinals, $(\mathfrak{J}^*)^\alpha$ is well-founded in M iff it is well-founded. Any path through $(\mathfrak{J}^*)^\alpha$ gives the appropriate β; completing the proof.

THEOREM 5. *If $\phi(\alpha)$ is any Σ^1_3 formula and $\mathfrak{J}^*$ is its associated tree, $\phi(\alpha)$ is equivalent to $\phi(\alpha)^{(L(\mathfrak{J}^*,\alpha))}$.*

We recall that a homogeneous Boolean algebra is a Boolean algebra in which the only elements invariant under all automorphisms are 0 and 1. A homogeneous Boolean extension of set theory is a Boolean extension of set theory using a homogeneous Boolean algebra. It is a basic fact of set theory that in a homogeneous extension all ordinal definable sets are standard [11].

THEOREM 6. *In any homogeneous Boolean extension of set theory for which κ*

is still a measurable cardinal, any Σ_3^1 statement has the same truth values as it has in the standard universe.

PROOF. By our stipulation of §2, the $\mathfrak{J}^*$ defined in the Boolean extension is ordinal definable; but in any homogeneous Boolean extension all ordinal definable sets are standard. Thus, using the standard universe of set theory as our transitive model of Theorem 4, the conclusion follows instantly.

We now turn out attention to a case of a Σ_3^1 formula which does not relativize. Solovay has shown that if there are two measurable cardinals and if μ is a measure on the smaller one, then there is a Σ_3^1 statement true in the universe and false in L_μ; viz. the assertion of the existence of the set he calls $0^\dagger$ [9, D 2040]. Thus the tree $\mathfrak{J}^*$ associated with this formula cannot be in L_μ and so there must be some part of the definition of $\mathfrak{J}^*$ which does not properly relativize to L_μ. This offending member can only be the function λ. That is to say there must be functions which are assigned different ordinals when the definition is performed inside L_μ and outside L_μ.

DEFINITION. If M is a transitive model for set theory, a *new ordinal* over M is a function f from $\kappa^{[n]}$ into the ordinals such that for every function g in M, $f \neq g$ almost everywhere.

We hope that we have convinced the reader of the truth of the following two theorems.

THEOREM 7. *If there are two measurable cardinals and if μ is a measure on the smaller, then there are new ordinals over L_μ.*

THEOREM 8. *If ϕ is any Σ_3^1 statement and if M is any transitive model for set theory together with the statement "κ is a measurable cardinal" and admitting no new ordinals, then ϕ is equivalent to $\phi^{(M)}$.*

We can now prove some classical properties of Σ_3^1 sets. Recall that a perfect set is a closed set with no isolated points. We say that a *perfect tree* is a tree of sequence numbers with no end nodes and no isolated branches. It is not hard to see that the perfect sets of reals are exactly those sets which can be represented as the set of paths through a perfect tree. Since all perfect sets have the cardinality of the continuum, it seemed in the early days of set theory that a promising program for proving the continuum hypothesis was to show that any uncountable set of reals contained a perfect subset. Some progress was made along these lines; Cantor showed that it held for closed sets and it was soon shown to hold for analytic sets [4, p. 386]. However, Luzin used the axiom of choice to exhibit a set such that neither it nor its complement contained a perfect set [5], and Gödel showed that with the assumption $V = L$ the argument could be used to produce a Δ_2^1 set with this pathological property [2, p. 67]. It is not hard to produce a Π_1^1 set in one-to-one correspondence with the set of constructible reals which can be shown using forcing methods not to contain a perfect subset [15]. What can be shown is that any Π_1^1 and hence Σ_2^1 set which contains a non-constructible element, contains a perfect subset [15]. Thus the assumption that there are only countably many constructible reals is needed in order to carry the program through to Σ_2^1. The next theorem is proven in [7].

THEOREM 9. *If A is any Π_2^1 set and $\mathfrak{J}^*$ is its associated tree and if A contains an element not in $L(\mathfrak{J}^*)$, then A contains a perfect subset.*

THEOREM 10. *If the Σ_3^1 set A is given by the formula $\exists\beta\psi(\alpha,\beta)$ where ψ is Π_2^1 and $\mathfrak{J}^*$ is the tree associated with ψ and if there are only countably many reals in $L(\mathfrak{J}^*)$, then A is Lebesgue measurable and has the property of Baire.*

PROOF. We have shown that $\alpha \in A$ is equivalent to

$$L(\mathfrak{J}^*, \alpha) \models \exists\beta\psi(\alpha,\beta)$$

Since the reals of $L(\mathfrak{J}^*)$ are assumed to be countable Solovay has proven that any set defined in this form is Lebesgue measurable and has the property of Baire [17], [18]. Indeed Solovay's proof allows the use of any formula of set theory in place of $\exists\beta\psi(x,\beta)$, provided the formula is used in the above format.

REFERENCES

1. J. W. Addison, *Some consequences of the axiom of constructibility*, Fund. Math. **46** (1959), 337-357.

2. K. Gödel, *The consistency of the continuum hypothesis* (2nd printing), Princeton Univer-ity Press, Princeton, N. J., 1940.

3. C. Kuratowski, *Ensembles projectifs et ensembles singuliers*, Fund. Math. **35** (1948), 131-140.

4. C. Kuratowski, *Topologie*, vol. 1, Panstowe Wydawnictwo Naukowe, Warsaw, 1958.

5. N. Lusin, *Sur L'existence d'un ensemble non dénorable qui est de première catégorie dans tout ensemble parfait*, Fund. Math. **2** (1921), 155–157.

6. R. Mansfield, *The theory of Σ_2^1 sets, doctoral disseratation*, Stanford University, 1969.

7. R. Mansfield, *Perfect subsets of definable sets of reals* Pacific J. Math. (to appear).

8. D. A. Martin and R. M. Solovay, *A basis theorem for Σ_3^1 sets of reals*, Ann of Math. **89**, (1969), 138–160.

9. A. R. D. Mathias, *A survey of recent results in set theory*, Proc. Sympos. Pure Math., Vol. 13, Amer. Math. Soc., Providence, R.I. (to appear).

10. G. Paris, doctoral dissertation, Manchester University, 1969.

11. D. Scott and R. M. Solovay, *Boolean valued models for set theory*, Proc. Sympos. Pure Math. Vol. **13**, Amer. Math. Soc., Providence, R.I. (to appear).

12. J. R. Shoenfield, *Mathematical Logic*, Addison Wesley Pub. Co., Menlo Park, Calif., 1967.

13. J. R. Shoenfield, *The problem of predicativity*, Essays on the Foundations of Mathematics pp. 132–139, Magnes Press, Jerusalem, 1961.

14. J. Silver, *Some applications of model theory to set theory*, doctoral dissertations, Univ. of California, Berkeley, 1966.

15. R. M. Solovay, *The cardinality of Σ^1_2 sets*, Foundations of Mathematics, Springer-Verlag, New York, 1968.

16. R. M. Solovay, *A non-constructible Δ^1_3 set of integers*, Trans. Amer. Math. Soc. **127** (1967), 50-75.

17. R. M. Solovay, Σ^1_2 *subsets of R are Lebesgue mesurable* (to appear).

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